Pulse dynamics in mode-locked lasers: relaxation oscillations and frequency pulling

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Abstract: A theoretical description of the pulse dynamics in a mode-locked laser including gain dynamics is developed. Relaxation oscillations and frequency pulling are predicted that influence the pulse parameters. Experimental observations of the response of a mode-locked Ti:sapphire laser to an abrupt change in the pump power confirm that the predicted behavior occurs. These results provide a framework for understanding the effects of noise on the spectrum of the laser.

OCIS codes: (140.4050) Mode-locked lasers.

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1. Introduction

Remarkable advances in the ability to measure and control the phase evolution of the pulses produced by mode-locked lasers have occurred over the last few years [1]. These advances have revolutionized optical frequency metrology [2], enabled optical atomic clocks [3, 4] and impacted high field physics [5]. Continued progress in these new applications of mode-locked lasers requires improvement in our understanding of their dynamics. Understanding the interplay between the pulse parameters is key to the development of control strategies and, more fundamentally, to understanding the ultimate stability limits due to quantum fluctuations.

The sensitivity of the pulse-to-pulse change in the carrier-envelope phase to the intensity was discovered [6] and exploited for control of the offset frequency [7, 8] early in the development of femtosecond comb technology. However it was clear that the effect was not simply due to the Kerr effect. There are contributions from changes in the center frequency [6, 8], which depends

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on the intensity through frequency pulling, contributions from changes in the pulse timing due to shock [9], and modification due to dispersion management in the laser [10]. These studies did not develop an understanding of the dynamical aspects of these effects, while experimental results only distinguished slow thermal effects from faster non-thermal ones [8].

An accurate theoretical description of the pulse dynamics will aid efforts to improve the feedback control of mode-locked lasers because the dynamics become part of the transfer function of the servo loop. Furthermore, such a theoretical description will allow the effect of noise on the frequency comb to be analyzed. Currently, technical noise dominates, although it can in principle be eliminated [11]. The fundamental limits will be set by amplified spontaneous emission (ASE), which cannot be eliminated [12]. Here, we focus on establishing the basis from which the effects of noise can be calculated. Experimental characterization of the dynamics show that the dynamics of the gain medium must also be included.

2. Background

Efforts to describe the dynamics of a mode-locked laser have been based on an approach described in a seminal paper by Haus and Mecozzi [13]. Assuming that only one polarization state plays a role and that the change of the pulse per round trip is small, so that one can replace the discrete laser components with continuous approximations, Haus and Mecozzi [13] obtained a master equation

$$T_R \frac{\partial u(T,t)}{\partial T} = \left[-i\theta_{\mathrm{sl}} + i\frac{D}{2}\frac{\partial^2}{\partial t^2} - l + g\left(1 + \frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t^2}\right) + (i\gamma + \delta)|u(T,t)|^2 \right] u(T,t), \quad (1)$$

where T_R is the laser cavity round trip time, u is the complex field envelope, normalized so that $|u|^2$ equals the instantaneous power, T is a slow time corresponding to $z/\overline{v}_g$, the folded distance along the laser divided by the average group velocity, and t is fast (retarded) time. We have explicitly added a phase slip θ_{sl} to the Haus-Mecozzi equations, so that the phase slip of u per round trip corresponds to the actual value at low power, where nonlinear effects can be neglected. Haus and Mecozzi assume that the dispersion $D = -\beta'' L$ is constant. We use β'' to designate the usual chromatic dispersion and L to designate the roundtrip length of the laser. Finally, g and l are linear gain and loss per round trip at the central frequency of the laser pulse, Ω_g is the gain bandwidth, γ is the Kerr coefficient, and δ is the fast saturable loss (or gain) parameter, which in a Ti:sapphire laser arises from the Kerr lens effect. We note that we have changed the definitions of D, γ, and δ from those of Haus and Mecozzi to bring the notation into closer alignment with the majority of the literature on optical solitons. However, like Haus and Mecozzi, we use T/T_R and not propagation length z as an independent variable. Consequently, the coefficient γ has units of inverse power, in contrast to the usual case in optical fiber solitons, where it has units of inverse power $\times$ inverse length.

Equation (1) applies in the slowly-varying envelope limit, in which the bandwidth of the laser pulse is small compared to its central frequency. This reference frequency ω_0, usually referred to as the angular carrier frequency, is removed from $u(T,t)$, so that its spectrum is shifted in the frequency domain towards zero frequency by this amount and is located in the neighborhood of zero frequency. A complete derivation of Eq. (1) in the context of optical fibers can be found in Refs. [14, 15]. While an analogous complete derivation for passively modelocked laser systems has yet to be published, the process is essentially the same. Because of group velocity dispersion, which is required to obtain a modelocked pulse, the pulse's round trip time only equals the round trip linear dispersive delay at one frequency. In order for Eq. (1) to hold and for the modelocked pulse intensity at equilibrium to be stationary at every point in t as a function of T, we must choose ω_0 equal to this frequency. Otherwise, a group velocity term

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proportional to $\partial u/\partial t$ with a real coefficient must be added to Eq. (1), or the pulse's central time changes at a constant rate as a function of T. There is some freedom in our choice of $\varpi_{\rm eq}$, the equilibrium pulse central frequency. In the laser system that we are considering, it is convenient to choose $\varpi_{\rm eq} = \omega_0$.

Equation (1) must be supplemented by an equation relating the slow saturable gain g to the pulse energy w, for which Haus and Mecozzi take

$$g = \frac{g_0}{1 + w/P_s T_R}, \tag{2}$$

where g_0 is the unsaturated gain and P_s is the saturation power. The soliton limit corresponds to $g, l \ll 1$, $\delta \ll \gamma$, and $g/\Omega_g^2 \ll |D|$. In this limit, Eq. (1) reduces at lowest order to the nonlinear Schrödinger equation, and both the linear and nonlinear gain and loss contributions appear as perturbations.

The pulses are characterized by four parameters: the pulse energy $w = w_{\rm eq} + \Delta w$, the central frequency $\varpi = \varpi_{\rm eq} + \Delta\varpi$, the central pulse time $\tau = \tau_{\rm eq} + \Delta\tau$, and the phase $\theta = \theta_{\rm eq} + \Delta\theta$, where $w_{\rm eq}$, $\varpi_{\rm eq}$, $\tau_{\rm eq}$, and $\theta_{\rm eq}$ are the equilibrium values of these quantities, while Δw, $\Delta\varpi$, $\Delta\tau$, and $\Delta\theta$ are their changes. Since the system is invariant under time and phase translations, we may without loss of generality choose $\tau_{\rm eq} = 0$ and $\theta_{\rm eq} = -\theta_{\rm ceo}(\omega_0)T/T_R$, where $\theta_{\rm ceo}$ is the carrier envelope offset phase shift per round trip in the laser at $\omega = \omega_0$. We note that $\theta_{\rm ceo} \neq \theta_{\rm sl}$ in general because of the nonlinear phase shift. This choice of phase, which is standard [13, 16, 17], greatly simplifies perturbation theory. We note for reference that Haus and Mecozzi's p corresponds to $-\Delta\varpi$.

In a laser with only one pulse in the cavity, the optical comb line shape is completely determined by the evolution of the four pulse parameters. One can use Eq. (1) and Eq. (2) to derive a linear equation governing their evolution including noise,

$$\frac{d\mathbf{v}}{dT} = -\mathsf{A}\cdot\mathbf{v} + \mathbf{S}, \tag{3}$$

where $\mathbf{v} = (\Delta w, \Delta\varpi, \Delta\tau, \Delta\theta)^t$ is the vector of the changes in the four pulse parameters (superscript t denotes the transpose). The quantity A is the 4×4 matrix of the constant coefficients that govern the linear response of each parameter to changes in either itself or the other parameters, while $\mathbf{S}$ is the vector of noise sources [13], which accounts for technical and quantum noise. One may now go on to calculate the timing and phase jitter [13] and the line widths [18, 19].

At this point, we confront the difficulty that the elements of A, which depend on both the parameters in Eq. (1) and the pulse shape (hyperbolic-secant for constant D, but closer to Gaussian when the system is dispersion-managed), are known at best qualitatively. This difficulty is particularly acute for A_{ww}, which depends on the unsaturated gain, the saturation power, and the nonlinear Kerr coefficient — none of which are easily measurable. Indeed, in Ti:sapphire lasers with soft Kerr lens modelocking, the variation of the fast saturable gain with power is not linear as implied by Eq. (1) [20]. Moreover, Eq. (1) implies that the frequency pulling coefficients $A_{\varpi x}$ (where x is g or w) are zero, which the experiments to be described shortly show is not the case. Thus, an accurate calculation of the line shape based on this approach is not possible.

To overcome this difficulty, we directly measure the elements of the matrix A. These experiments not only provide the data that are needed to calculate the fundamental line shapes; in addition, they allow us to determine the values of important physical parameters that are needed to calculate the noise sources $\mathbf{S}$ but are difficult to directly measure. These measurements yield important insights into the laser behavior that point the way toward more complete and accurate underlying physical models than Eq. (1).

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3. Experiment

To measure the elements of matrix A, we abruptly changed the pump power and measured the output signal power, the fluorescence from the laser crystal, and the spectral shift in a Kerr-lens mode-locked Ti:sapphire laser with 15 fs FWHM pulses, a repetition rate of 94 MHz, and an intracavity pulse energy in the neighborhood of 55 nJ. We show typical results in Fig. 1. The emitted fluorescence approximates the gain of the laser pulse, but is not exactly proportional to it; in particular, the phase of the fluorescence is not the same as the phase of the gain. The pump power was modulated by an acousto-optic modulator with a switching time of roughly 200 ns. The modulation was a square wave with a period of 10 ms and a depth of about 1%. We averaged approximately 100 traces using a digital oscilloscope. To measure $\Delta\varpi$, the laser intensity, spectrally resolved with a monochromator, was measured using a photodiode. We measure time traces for a range of wavelengths, covering the entire laser spectrum. The central frequency is approximated as the centroid of the frequency spectrum.

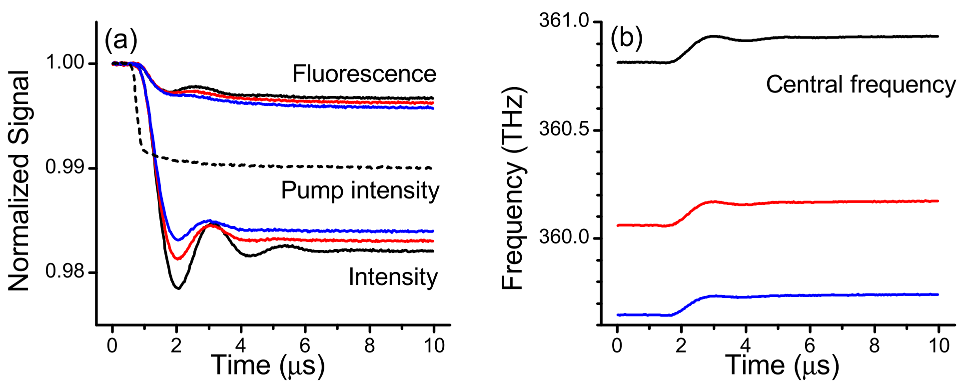

Fig. 1. Experimental data showing the (a) laser intensity and fluorescence and (b) central frequency as a function of time following a sudden drop in the pump power. Curves are for pump powers of 5.1 W (black), 5.3 W (red) and 5.5 W (blue). The curves (a) are normalized to the value prior to the step in pump power. In (a) the pump intensity is also shown (dashed line).

Two salient points are evident. First, there can be long-lived relaxation oscillations. Spontaneous oscillations in the intensity and central frequency have been observed previously [21], although the spontaneous nature of them prevented extensive analysis. Equation (2) is inadequate to describe this behavior and must be replaced with the dynamical equation

$$\frac{\mathrm{d}g}{\mathrm{d}T} = \frac{g_0 - g}{\tau_f} - \frac{1}{\tau_f}\frac{gw}{P_s T_R}, \tag{4}$$

where τ_f is the fluorescence lifetime of the medium [22], and we are assuming that the Ti:sapphire crystal may be treated as an ideal four-level system, so that $g \propto N_2$, the population of the upper lasing level. As a consequence, Eq. (3) as a 4-dimensional system is incomplete. It must be replaced by a similar 5-dimensional system in which the vector $\mathbf{v}$ becomes $(\Delta w, \Delta\varpi, \Delta\tau, \Delta\theta, \Delta g)^t$, where Δg is the change in the gain, and A becomes a 5×5 matrix. Second, there is significant frequency pulling. Frequency pulling is not included in Eq. (1), which must be appropriately modified.

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These measurements have significant theoretical implications. We can infer A_{wx}, A_{gx}, and $A_{\varpi x}$, where $x = w$, ϖ, τ, θ, or g. We can also infer N_2, the number of atoms in the upper lasing level. This parameter is important in determining **S**. Finally, we may infer minimal modifications to (1) that incorporate frequency pulling, although these modifications are not unique. In principle, if the gain and loss as a function of frequency were known, we could directly calculate the frequency-pulling coefficients. However, the gain and loss as a function of frequency are difficult to measure and not well known.

Measurements of $A_{\tau x}$ and $A_{\theta x}$ require a different, more sophisticated experimental technique that will be the subject of a later publication. In related work, Jiang, *et al.* [23] discussed a 5×5 system in the context of semiconductor lasers, and Matos, *et al.* [24] discussed the transient gain dynamics in a Ti:sapphire laser and the importance of accounting for the variation of the cavity lifetime τ_{ph} with w. Paschotta [25, 26] has described a computational approach for calculating the noise statistics in which the discrete laser components are kept, and the evolution of the dispersive continuum as well as the pulse parameters is followed.

4. Gain and intensity dynamics

On the long time scale, the evolution of the pulse energy is given by

$$\frac{\mathrm{d}w}{\mathrm{d}T} = -\frac{w}{\tau_{\mathrm{ph}}(w)} + \frac{2gw}{T_R}. \tag{5}$$

While τ_{ph} can be related to the parameters of an underlying physical model such as Eq. (1), it is not useful to do so at this point. We note however that τ_{ph} includes nonlinear contributions from the fast saturable gain as well as linear contributions from the slow gain and loss. Linearizing Eqs. (4) and (5) about equilibrium values g_{eq} and w_{eq}, and using the relationship $1/\tau_{\mathrm{ph}} = 2g_{\mathrm{eq}}/T_R$, which comes from the equilibrium solution to Eq. 5, we find equations of motion for the deviations from equilibrium

$$\begin{aligned} \frac{\mathrm{d}\Delta w}{\mathrm{d}T} + A_{ww}\Delta w + A_{wg}\Delta g &= 0, \\ \frac{\mathrm{d}\Delta g}{\mathrm{d}T} + A_{gw}\Delta w + A_{gg}\Delta g &= \frac{\Delta g_0}{\tau_f}, \end{aligned} \tag{6}$$

where $A_{ww} = (\mathrm{d}\tau_{\mathrm{ph}}^{-1}/\mathrm{d}w)w_{\mathrm{eq}}$, $A_{wg} = -2w_{\mathrm{eq}}/T_R$, $A_{gw} = (g_{\mathrm{eq}}/\tau_f)(1/P_s T_R)$, and $A_{gg} = (1 + w_{\mathrm{eq}}/P_s T_R)(1/\tau_f)$. All other A_{wx} and A_{gx} equal zero.

The abrupt change in pump power causes a sudden change in the unsaturated gain Δg_0, leading to the damped oscillations shown in Fig. 1. The damping rate is given by $\alpha = (A_{ww} + A_{gg})/2$ and the oscillation frequency is given by $\omega_{\mathrm{osc}}^2 = -A_{wg}A_{gw} + A_{ww}A_{gg} - \alpha^2$. We show measured ω_{osc} and α in Fig. 2 as a function of the pump power for both modelocked and continuous wave (cw) operation. These values are obtained by explicitly solving Eq. (6) and fitting ω_{osc} and α to the analytical form, shown in Appendix A, using the method of least squares. As the pump power varies between 4.7 and 5.4 W, we find that that ω_{osc} varies from 2.6×10^6 rad/s to 2.8×10^6 rad/s and α varies from $0.2 \times 10^6\ \mathrm{s}^{-1}$ to $1.0 \times 10^6\ \mathrm{s}^{-1}$ for the modelocked operation. The variation of ω_{osc} in the cw case is qualitatively similar to the variation in the modelocked case, but the variation of α differs significantly, being far more gradual.

It is useful to rewrite Eq. (6) in terms of the photon number $N_{\mathrm{ph}} = w/\hbar\varpi$ and the number of atoms in the upper lasing level $N_2 = (V/\sigma l_g)g$, where $\hbar$ is Planck's constant, V is the effective

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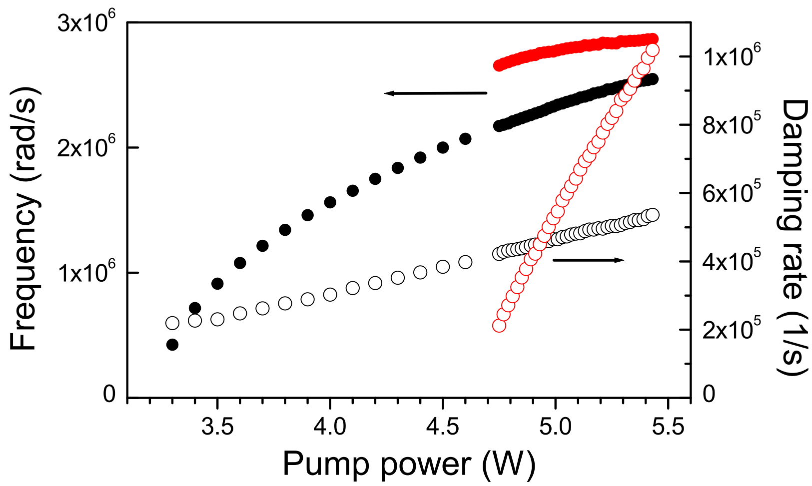

Fig. 2. Measured oscillation frequency (filled circles, left axis) and damping rate (open circles, right axis) as function of pump power for mode-locked (red) and cw (black) operation of the laser.

gain volume, l_g is the effective gain length, and σ is the gain cross section. We then find

$$\begin{aligned}\frac{\mathrm{d}\Delta N_2}{\mathrm{d}T} &= -\left(\frac{1}{\tau_f}+\frac{1}{\tau_{\text{ph}}}\frac{N_{\text{ph,eq}}}{N_{2,\text{eq}}}\right)(\Delta N_2-\Delta N_{20})+\frac{1}{\tau_{\text{ph}}}\Delta N_{\text{ph}},\\ \frac{\mathrm{d}\Delta N_{\text{ph}}}{\mathrm{d}T} &= \frac{\mathrm{d}\tau_{\text{ph}}^{-1}}{\mathrm{d}N_{\text{ph}}}\Delta N_{\text{ph}}-\frac{1}{\tau_{\text{ph}}}\frac{N_{\text{ph,eq}}}{N_{2,\text{eq}}}\Delta N_2,\end{aligned} \tag{7}$$

where ΔN_{20} indicates the abrupt change in N_2 corresponding to Δg_0, and $N_{\text{ph,eq}}$ and $N_{2,\text{eq}}$ indicate the equilibrium photon number and upper state population, respectively. Ignoring the damping contributions, we find that $-A_{wg}A_{gw}=(1/\tau_{\text{ph}})^2(N_{\text{ph,eq}}/N_{2,\text{eq}})$, from which we obtain the important result,

$$\frac{N_{2,\text{eq}}}{N_{\text{ph,eq}}}=\frac{1}{-A_{wg}A_{gw}\tau_{\text{ph}}^2}=13, \tag{8}$$

where we used the experimental value for the cavity lifetime $\tau_{\text{ph}}=0.1\ \mu$s, and we may infer $(-A_{wg}A_{gw})^{1/2}\simeq 2.8\times 10^6$ rad/s by using this value and showing that it produces the variation of ω_{osc} shown in Fig. 2. Using $w_{\text{eq}}=55$ nJ and $\varpi_{\text{eq}}=2.3\times 10^{15}$ rad/s, we obtain $N_{\text{ph,eq}}=2.3\times 10^{11}$ and $N_{2,\text{eq}}=2.9\times 10^{12}$. Using the expression $A_{gg}=(1/\tau_f)+(1/\tau_{\text{ph}})(N_{\text{ph,eq}}/N_{2,\text{eq}})$ and the measured value $\tau_f=2.5\ \mu$s, we find $A_{gg}=1.2\times 10^6\ \text{s}^{-1}$, and we also find that $A_{ww}=2\alpha-A_{gg}$ varies from $-0.8\times 10^6\ \text{s}^{-1}$ at a pump power of 4.7 W to $0.8\times 10^6\ \text{s}^{-1}$ at a pump power of 5.4 W. We note that the laser remains stable when A_{ww} becomes negative, although the relaxation oscillations become long-lived. This behavior is very different from the cw behavior shown in Fig. 2 and indicates that relaxation oscillations may be important in modelocked lasers even when they are not important in the same laser generating cw light. Using $(-A_{wg}A_{gw})^{1/2}=2.8\times 10^6$ rad/s and the relationship $\omega_{\text{osc}}^2=-A_{wg}A_{gw}+A_{ww}A_{gg}-\alpha^2$ implies that ω_{osc} varies from 2.6×10^6 rad/s to 2.8×10^6 rad/s, consistent with Fig. 2. Using the relation $A_{wg}=-2w_{\text{eq}}/T_R$ and the measured value $w_{\text{eq}}=55$ nJ, we infer $A_{wg}=-11\ \text{J}\,\text{s}^{-1}$ and $A_{gw}=7.1\times 10^{11}\ \text{J}^{-1}\text{s}^{-1}$.

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5. Frequency pulling

The frequency pulling shown in Fig. 1 is governed by the equation

$$\frac{d\Delta\varpi}{dT} + A_{\varpi w}\Delta w + A_{\varpi g}\Delta g + A_{\varpi\varpi}\Delta\varpi = 0, \tag{9}$$

where $A_{\varpi\tau} = A_{\varpi\theta} = 0$. We determine the $A_{\varpi x}$ from the experimental data by using the method of least squares to fit the analytical form of the solution to Eq. (9), shown in Appendix A. We find $A_{\varpi\varpi} = 1.2\times10^6$ s^{-1}, $A_{\varpi g} = -3.0\times10^8$ THz/s, and $A_{\varpi w} = 1.2\times10^5$ THz/(nJ s) at 5.1 W pump power.

The physical origin of the gain-induced frequency pulling is the finite bandwidth of the loss [13]. As the gain changes, the equilibrium frequency changes and the pulse's central frequency is pulled toward the new equilibrium. The physical origin of the pulse-energy-induced frequency pulling is asymmetry in the gain and loss. As the pulse's energy changes, its bandwidth also changes and the asymmetry then shifts the location of the equilibrium. Neither of these effects are included in Eq. (1). We may include them in the limit that the pulse bandwidth is small compared to the gain and loss bandwidths by adding a third order Taylor expansion of the gain and loss profiles to Eq. (1). In the frequency domain, the gain $\tilde{g}(\omega)$ and the loss $\tilde{l}(\omega)$ become

$$\begin{aligned}\tilde{g}(\omega) &= g^{(0)} + g^{(1)}\omega + \frac{1}{2}g^{(2)}\omega^2 + \frac{1}{6}g^{(3)}\omega^3,\\ \tilde{l}(\omega) &= l^{(0)} + l^{(1)}\omega + \frac{1}{2}l^{(2)}\omega^2 + \frac{1}{6}l^{(3)}\omega^3,\end{aligned} \tag{10}$$

where ω indicates the change in frequency with respect to the carrier frequency ω_0, so that the actual frequency is given by $\omega_0 + \omega$. The coefficients $g^{(m)}$ and $l^{(m)}$ indicate the m^{th} derivatives of the gain and loss with respect to frequency. With this expansion, the operator $-l + g[1 + (1/\Omega_g^2)\partial^2/\partial t^2]$ is replaced by $[g^{(0)} - l^{(0)}] + i[g^{(1)} - l^{(1)}]\partial/\partial t - (1/2)[g^{(2)} - l^{(2)}]\partial^2/\partial t^2 - i(1/6)[g^{(3)} - l^{(3)}]\partial^3/\partial t^3$. Changes in the gain will produce an additional contribution of $i\Delta g^{(1)}\partial/\partial t$ at lowest non-trivial order. Identifying $(1/2)[g^{(2)} - l^{(2)}] \equiv \Omega_g^2$, we see that we are adding an additional perturbation to Eq. (1) of the form $P[u] = i[g^{(1)} - l^{(1)}]\partial u/\partial t - i[g^{(3)} - l^{(3)}]\partial^3 u/\partial t^3 + i\Delta g^{(1)}\partial u/\partial t$. Since our system is dispersion-managed, the pulses are nearly Gaussian in shape. So, it is appropriate to use a perturbation expansion based on Gaussian-shaped pulses. We show in Appendix B that

$$A_{\varpi w} = -\frac{3.23}{T_R}\left[g^{(3)} - l^{(3)}\right]\frac{1}{t_{\text{FWHM}}^4 w_{\text{eq}}}, \qquad A_{\varpi g} = -\frac{2.77}{T_R}\frac{g^{(1)}}{g^{(0)}}\frac{1}{t_{\text{FWHM}}^2}, \tag{11}$$

where t_{FWHM} is the full width half maximum pulse duration of 15 fs. Assuming with Haus and Mecozzi [13] a nominal gain bandwidth $\Omega_g = 1.6\times10^{15}$ rad/s and using the experimentally determined value of $A_{\varpi g}$, we obtain $g^{(1)}\Omega_g/g^{(0)} = 0.42$, which is consistent with the loss bandwidth peak lying below the gain bandwidth peak. Using the experimental value of $A_{\varpi w}$, we also find $[g^{(3)} - l^{(3)}]\Omega_g^3 = -4.5$, indicating a substantial asymmetry over the nominal gain bandwidth. We note that these inferences are not unique, since they assume that higher orders of the Taylor expansion of the gain and loss curves do not contribute. Nonetheless, these results show that it is possible to infer corrections to Eq. (1) from the measurements of A.

6. Conclusion

In summary, we have experimentally measured some elements in the matrix A that describes the dynamics of the pulse parameters in a Ti:sapphire mode-locked laser. We have shown that it

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is necessary to take into account the gain dynamics and frequency pulling. Once that is done, it is possible to measure all coefficients of the form A_{wx}, A_{gx}, and $A_{\varpi x}$. One may use these results to infer N_2, which is needed to determine the strengths of the quantum noise sources. The next step, measurement of $A_{\tau x}$ and $A_{\theta x}$, is underway. This information can be used with a full perturbation theory for this system with realistic pulse shapes that take into account dispersion management to yield a complete calculation of the entire line shape.

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Appendix A. Solutions to Eqs. (6) and (9)

We may obtain the solution to Eq. (6) using the method of variation of parameters, assuming that Δg_0 changes instantaneously at $t=0$ to its final value. The solution is:

$$\begin{aligned}\Delta w &= -\frac{A_{wg}}{\omega_{\mathrm{osc}}^2+\alpha^2}\frac{\Delta g_0}{\tau_f}\left\{1-e^{-\alpha T}\left[\cos(\omega_{\mathrm{osc}}T)+\frac{\alpha}{\omega_{\mathrm{osc}}}\sin(\omega_{\mathrm{osc}}T)\right]\right\},\\ \Delta g &= \frac{A_{ww}}{\omega_{\mathrm{osc}}^2+\alpha^2}\frac{\Delta g_0}{\tau_f}\left\{1-e^{-\alpha T}\left[\cos(\omega_{\mathrm{osc}}T)+\frac{\alpha}{\omega_{\mathrm{osc}}}\sin(\omega_{\mathrm{osc}}T)\right]\right\}\\ &\quad+\frac{1}{\omega_{\mathrm{osc}}}\frac{\Delta g_0}{\tau_f}e^{-\alpha T}\sin(\omega_{\mathrm{osc}}T),\end{aligned} \tag{A.1}$$

which may be verified by substitution.

We obtain the solution to Eq. (9) by first substituting Eq. (A.1) into Eq. (9). We next rewrite Eq. (9) in the form,

$$\begin{aligned}\frac{\mathrm{d}\Delta\varpi\exp(A_{\varpi\varpi}T)}{\mathrm{d}T} &= \frac{(A_{\varpi w}A_{wg}-A_{\varpi g}A_{ww})}{\omega_{\mathrm{osc}}^2+\alpha^2}\frac{\Delta g_0}{\tau_f}\\ &\quad\times\Big\{\exp(A_{\varpi\varpi}T)-\exp\left[-(\alpha-A_{\varpi\varpi})T\right]\cos(\omega_{\mathrm{osc}}T)\\ &\qquad-\frac{\alpha}{\omega_{\mathrm{osc}}}\exp\left[-(\alpha-A_{\varpi\varpi})T\right]\sin(\omega_{\mathrm{osc}}T)\Big\}\\ &\quad-A_{\varpi g}\frac{1}{\omega_{\mathrm{osc}}}\frac{\Delta g_0}{\tau_f}\exp\left[-(\alpha-A_{\varpi\varpi})T\right]\sin(\omega_{\mathrm{osc}}T).\end{aligned} \tag{A.2}$$

Defining $\bar{\alpha}\equiv\alpha-A_{\varpi\varpi}$, we integrate this equation to obtain

$$\begin{aligned}\Delta\varpi &= \frac{(A_{\varpi w}A_{wg}-A_{\varpi g}A_{ww})}{\omega_{\mathrm{osc}}^2+\alpha^2}\frac{\Delta g_0}{\tau_f}\Big\{\frac{1}{A_{\varpi\varpi}}\left[1-\exp(-A_{\varpi\varpi}T)\right]\\ &\qquad-\frac{1}{\omega_{\mathrm{osc}}}\frac{\omega_{\mathrm{osc}}^2-\alpha\bar{\alpha}}{\omega_{\mathrm{osc}}^2+\bar{\alpha}^2}\exp(-\alpha T)\sin(\omega_{\mathrm{osc}}T)\\ &\qquad-\frac{\alpha+\bar{\alpha}}{\omega_{\mathrm{osc}}^2+\bar{\alpha}^2}\left[\exp(-A_{\varpi\varpi}T)-\exp(-\alpha T)\cos(\omega_{\mathrm{osc}}T)\right]\Big\}\\ &\quad+A_{\varpi g}\frac{1}{\omega_{\mathrm{osc}}}\frac{\Delta g_0}{\tau_f}\Big\{\frac{\bar{\alpha}}{\omega_{\mathrm{osc}}^2+\bar{\alpha}^2}\exp(-\alpha T)\sin(\omega_{\mathrm{osc}}T)\\ &\qquad-\frac{\omega_{\mathrm{osc}}}{\omega_{\mathrm{osc}}^2+\bar{\alpha}^2}\left[\exp(-A_{\varpi\varpi}T)-\exp(-\alpha T)\cos(\omega_{\mathrm{osc}}T)\right]\Big\}.\end{aligned} \tag{A.3}$$

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Appendix B. Generalized Perturbation Theory

Perturbation analysis is a critical tool in the study of short-pulse lasers, and it has been applied to a wide variety of problems [19]. A key difficulty with the work to date is that it uses traditional soliton perturbation theory that assumes that at zero order the pulses are hyperbolic-secant shaped. In fact, virtually all of today's short-pulse laser systems are dispersion-managed, and the pulses are close to Gaussian shaped. Often, the traditional approach yields excellent agreement between theory and experiment [27]. However, there are notable exceptions, such as the calculation of the carrier-envelope phase shift in which the traditional approach yields qualitatively incorrect answers. (Compare [9] and [10].) This approach is unreliable and an approach that can incorporate the correct zero-order pulse shape is needed.

A key difficulty with developing perturbation theory for the pulses in dispersion-managed systems is that a zero-order averaged equation is not generally available. Haus, *et al.* [28] proposed an equation in which the time variable appears explicitly, but as Smith, *et al.* [29] first pointed out, this equation does not scale correctly with the pulse energy. The explicit appearance of the time variable breaks the physically-required zero-order time-invariance symmetry. Hence, this equation is not suitable as the starting point for a perturbation analysis. Ablowitz and Biondini [30] and Gabitov and Turitsyn [31] have derived integro-differential equations that are non-local in time. These can be used as the basis for a perturbation theory. However, they only apply when the map strength is large, *i.e.*, the variation of the dispersion is large compared to the average. Typically, the map strengths in laser systems are moderate, and the ratio of the variation to the average is close to 1. Moreover, one would like to have an approach that can be applied to computational studies like those of Paschotta [25, 26] that take into account the discreteness of the laser components. It is not at all clear that it is possible to obtain a closed-form zero-order equation in this case, while it is certainly possible to obtain computationally the equilibrium pulse shape. In this case, one would like to be able to use the computationally-determined pulse shape as the starting point for a perturbation analysis.

Here, we report that we have circumvented the difficulties just described and have developed a perturbation theory that allows one to use empirically- or computationally-derived pulse shapes even when there is no closed-form expression for the underlying averaged equations, as long as certain physical and mathematical assumptions — that are well supported by experimental, computational, and theoretical work to date — are obeyed. We will not discuss all the details here, reserving that for a later publication. However, all the assumptions that are used in the theoretical development are presented here, and the development is complete, albeit brief. Likewise, we do not give extensive examples of the theory's application here. Instead, we show how the apparatus of the theory can be used to extract the result on frequency pulling in the main text, which is all that is needed for this paper.

We begin by assuming that we have model equations that govern the light evolution in the cavity and are periodic in the folded time T, *i.e.*, $T_R \partial u(T,t)/\partial T = \mathscr{F}[u(T,t)]$. While Eq. (1) is an example of just such a model, the computational model of Paschotta [25, 26] is another example that consists of discrete portions concatenated together. We assume that in the absence of gain and loss, the model equations have a stationary short-pulse solution $u_0(T,t)$ that depends only on the four parameters w, ϖ, τ, and θ. Gain and loss are needed to set the equilibrium values of w and ϖ, and loss is needed to damp the continuum radiation, but in the dispersion-managed soliton regime in which virtually all of today's short-pulse lasers operate, it is appropriate to treat these terms perturbatively. In addition, we will assume that the underlying equations are time- and phase-invariant at zero order, so that $u_0(T,t;w,\varpi,\tau,\theta) = u_0(T,t-\tau;w,\varpi)\exp(i\theta)$. While it is not necessary for our development, we will also assume that at zero order the underlying equations are frequency-invariant at zero order. It is not difficult to find model systems with no gain or loss that violate this assumption. If

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higher-order dispersion for example is included in the zero-order system, then this assumption is not valid. However, this assumption holds at zero order in the dispersion-managed systems that are of interest to us here, and we find $u_0(T,t) = u_0(T,t-\tau;w)\exp[-i\Delta\varpi(t-\tau)+i\theta]$, where we have used the assumption $\omega_0 = \varpi_{\mathrm{eq}}$.

If we now write $u(T,t) = u_0(T,t) + \Delta u(T,t)$ and we linearize $\mathscr{F}[u]$ about the equilibrium (periodically stationary) solution $u_0(T,t)$, we obtain a linear Bloch-Floquet equation with periodically varying coefficients. Starting at any location in the laser, we may integrate this equation over one round trip and divide by T_R to obtain an averaged equation that governs the slow evolution and that may be written in the form

$$\frac{\partial \Delta u}{\partial T} = i\mathscr{M}[u_0]\Delta u + i\mathscr{N}[u_0]\Delta u^*, \tag{B.1}$$

where $\mathscr{M}$ and $\mathscr{N}$ are operators that may be non-local in time. In the case of the nonlinear Schrödinger equation with constant dispersion, $\mathscr{M} = (D/2)\partial^2/\partial t^2 + 2\gamma|u_0|^2$ and $\mathscr{N} = \gamma u_0^2$. In keeping with our assumption that the zero-order system has no gain or loss, we assume that $\mathscr{M}$ is a Hermitian operator and $\mathscr{N}$ is symmetric, by which we mean that given any $u(t)$ and $v(t)$,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, v^*(t_1)\mathscr{M}(t_1,t_2)u(t_2) &= \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, u(t_2)\mathscr{M}^*(t_2,t_1)v^*(t_1),\\ \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, v^*(t_1)\mathscr{N}(t_1,t_2)u^*(t_2) &= \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, u^*(t_2)\mathscr{N}(t_2,t_1)v^*(t_1).\end{aligned} \tag{B.2}$$

The connection to gain and loss of this assumption may not be entirely obvious to the reader, and it will be elucidated elsewhere in a full discussion of perturbation theory. However, the reader can easily verify that a vast number of equations with no gain and loss satisfy these conditions once linearized, including all equations of practical interest known to the authors, such as the nonlinear Schrödinger equation, the Ablowitz-Biondini equation [30] that governs dispersion-managed solitons, and the equation proposed by Haus, *et al.* [28]. Equations with any order of dispersion and arbitrary (non-singular) nonlinearities, *i.e.*, $|u|^2u \to F[|u|^2]u$, also satisfy these conditions once linearized.

We now define an inner product $\langle v|u\rangle = (1/2)\int_{-\infty}^{\infty} \mathrm{d}t\,(v^*u + vu^*)$ and note that if u satisfies Eq. (B.1), then $v = iu$ satisfies the dual equation

$$\frac{\partial v}{\partial T} = i\mathscr{M}[u_0]v - i\mathscr{N}[u_0]v^*. \tag{B.3}$$

Any solution to Eq. (B.1) may be written $\Delta u = u_w\Delta w + u_{\varpi}\Delta\varpi + u_\tau\Delta\tau + u_\theta\Delta\theta + \Delta u_c$, where $u_x = \partial u_0/\partial x$, $x = w$, ϖ, τ, or θ, and Δu_c is a dispersive wave continuum. We first have the important result that since the u_x satisfy Eq. (B.1), $v_x \equiv iu_x$ must satisfy the dual equations. Second, we may show $\langle v_x|\Delta u_c\rangle = 0$, using an approach that is analogous to the approach used in traditional soliton perturbation theory [32]. Thus by operating with the four $\langle v_x|$ on any perturbed equation, we may find the effect of the perturbation on the four soliton parameters, just as in traditional soliton perturbation theory.

Given an arbitrary pulse shape u_0, the $\langle v_x|u_y\rangle$ are non-zero in general for all combinations of x and y. However, in the special case in which the pulses are symmetric about $t-\tau$ and are entirely in one phase — as is the case for both standard and dispersion-managed solitons at their point of minimum compression — then we find $\langle v_x|u_y\rangle = 0$ when $x \neq y$. In practice, the point of minimum compression is arranged to be at the exit mirror of the laser, so that is the point at which the pulses are observed.

We now apply this general theoretical apparatus to Gaussian-shaped pulses, which computational and experimental studies have demonstrated is an excellent approximation to the

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true pulse shape. In principle, we could also use the exact, computationally-determined pulse shapes. We consider the perturbations that lead to frequency pulling,

$$P[u] = i\left[g^{(1)} - l^{(3)}\right]\frac{\partial u}{\partial t} - \frac{i}{6}\left[g^{(3)} - l^{(3)}\right]\frac{\partial^3 u}{\partial t^3} + i\Delta g^{(1)}\frac{\partial u}{\partial t}. \tag{B.4}$$

The equilibrium pulse shape is given by $u_0(t) = A\exp\left[-(t-\tau)^2/2t_p^2\right]\exp[-i\Delta\varpi(t-\tau)+i\theta]$. We must now relate A and t_p to the pulse energy w. Here, we may appeal to the computationally-determined relations [33, 34] in a system with two dispersive media of lengths L_1 and L_2 with dispersions β_1'' and β_2'' and an average dispersion $\beta_{\text{av}}'' = (\beta_1''L_1+\beta_2''L_2)/(L_1+L_2)$. It was shown that $w = (r/t_p)(1+s/t_p^4)$, where $r = 2|\beta_{\text{av}}''|(L_1+L_2)/\gamma$ and $s = 0.09|(\beta_1''-\beta_{\text{av}}'')L_1 - (\beta_2''-\beta_{\text{av}}'')L_2|^2$ [33, 34]. We also have $w = \sqrt{\pi}A^2 t_p$, which allows us to relate A to w. We now find,

$$\begin{aligned} u_w &= \frac{1}{w}\left[C_1(t_p) - C_2(t_p)\frac{(t-\tau)^2}{t_p^2}\right]u_0(t), \quad & u_\varpi &= -i(t-\tau)u_0(t), \\ u_\tau &= \frac{t-\tau}{t_p^2}u_0(t), & u_\theta &= iu_0(t), \end{aligned} \tag{B.5}$$

along with the duals,

$$\begin{aligned} v_w &= 2u_0(t), & v_\varpi &= -\frac{2i}{w}\frac{t-\tau}{t_p^2}u_0(t), \\ v_\tau &= \frac{2}{w}(t-\tau)u_0(t), & v_\theta &= \frac{2i}{w}\left[C_1(t_p) - C_2(t_p)\frac{(t-\tau)^2}{t_p^2}\right]u_0(t), \end{aligned} \tag{B.6}$$

where

$$C_1(t_p) = \frac{1+3s/t_p^4}{1+5s/t_p^4}, \qquad C_2(t_p) = \frac{1+s/t_p^4}{1+5s/t_p^4}. \tag{B.7}$$

The duals are chosen so that $\langle v_x|u_x\rangle = 1$. Operating on $P[u_0]$ with $\langle v_\varpi|$, we obtain

$$T_R\frac{\mathrm{d}\Delta\varpi}{\mathrm{d}T} = \left[g^{(1)} - l^{(1)}\right]\frac{1}{t_p^2} + \frac{1}{4}\left[g^{(3)} - l^{(3)}\right]\frac{1}{t_p^4} + \frac{\Delta g^{(1)}}{t_p^2}. \tag{B.8}$$

Because of the modelocked pulse's finite bandwidth and the assumed asymmetry of the gain, which appears in the third-derivative term, the pulse will not reside at the peak of the combined gain-loss curve, and its derivative is not zero when $\omega = 0$. In order to enforce the condition that $\mathrm{d}\Delta\varpi/\mathrm{d}T = 0$ at equilibrium, we must set $[g^{(1)} - l^{(1)}] = -[g^{(3)} - l^{(3)}]/4t_{p,\text{eq}}^2$, from which we find

$$\begin{aligned} T_R\frac{\mathrm{d}\Delta\varpi}{\mathrm{d}T} &= \frac{1}{4t_p^2}\left[g^{(3)} - l^{(3)}\right]\left(\frac{1}{t_{p,\text{eq}}^2} - \frac{1}{t_p^2}\right) + \frac{\Delta g^{(1)}}{t_p^2} \\ &= \frac{1}{2t_p^5}\left[g^{(3)} - l^{(3)}\right]\Delta t_p + \frac{\Delta g^{(1)}}{t_p^2}. \end{aligned} \tag{B.9}$$

Finally, using the relations $\Delta t_p = (\mathrm{d}t_p/\mathrm{d}w)\Delta w = -[(t_p/w)(1+s/t_p^4)/(1+5s/t_p^4)]$ and $\Delta g^{(1)} = (g^{(1)}/g^{(0)})\Delta g$, we conclude

$$A_{\varpi w} = -\frac{1}{2T_R t_p^4}\left[g^{(3)} - l^{(3)}\right]\frac{1+s/t_p^4}{1+5s/t_p^4}, \qquad A_{\varpi g} = -\frac{1}{T_R t_p^2}\frac{g^{(1)}}{g^{(0)}}. \tag{B.10}$$

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To arrive at the expressions in the main text, we substitute $t_p = t_{\mathrm{FWHM}}/[2\sqrt{\ln 2}] = 0.6006 t_{\mathrm{FWHM}}$. Using the expression from [33], $s/t_p^4 = 0.7|(\beta_1'' - \beta_{\mathrm{av}}'')L_1 - (\beta_2'' - \beta_{\mathrm{av}}'')L_2|^2/t_{\mathrm{FWHM}}^4$, as well as the result from [35] that for $t_{\mathrm{FWHM}} = 15$ fs, we have $|(\beta_1'' - \beta_{\mathrm{av}}'')L_1 - (\beta_2'' - \beta_{\mathrm{av}}'')L_2| = 60\ \mathrm{fs}^2$, we find $s/t_p^4 = 0.05$ and $(1+s/t_p^4)/(1+5s/t_p^4) = 0.84$.

We note that if the traditional perturbation theory is used, the coefficient 3.23 in the expression for $A_{\varpi w}$ becomes 3.00 and the coefficient 2.77 in the expression for $A_{\varpi g}$ becomes 2.07. So, as is often the case, the traditional theory agrees well with the generalized approach described here. However, there is no compelling reason to use the traditional approach. The generalized approach is no more difficult to apply than the traditional approach and makes use of the true zero-order pulse shapes with whatever exactitude the problem at hand requires.

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